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BIOGRAPHY.

HUBERT ANSON NEWTON.

BY PROFESSOR ANDREW W. PHILLIPS.

HUBERT ANSON NEWTON was born in Sherburne New York, March 19, 1830, and died at New Haven on the 12th day of August, 1896.* He graduated from Yale, taking the degree of A. B., in 1850, and spent the next two and one-half years in mathematical study. He became tutor at Yale in 1853 and on account of the sickness and subsequent death of Professor Stanley, the whole work of the department of mathematics devolved upon him from the first. In 1855 his great ability was recognized in his election, at the early age of twenty-five, to a full professorship of mathematics at Yale, the duties of which he assumed after spending a year of study in Europe, where, under the inspiration of Chasles, he became especially interested in the subject of Modern Higher Geometry. He carried on most vigorously work and studies in various lines in addition to the duties of his professorship. Sometimes it was a profound study in pure Mathematics, sometimes a rich contribution to the education of the public, and sometimes an original investigation in the field of Astronomy.

He published in 1857 a paper on the Gyroscope in the *American Journal of Science*, and soon after, a paper in the *Mathematical Monthly*, in which he seems to have been the first to apply the principle of inversion in the solution of the problem of constructing circles tangent to three given circles. He showed how deeply rooted in his mind were the ideas of the Modern Geometry in his

*Professor Newton was Vice President of the American Mathematical Society at the time of his death. EDITOR.

elaborate papers published in the same journal in 1861 on the geometrical construction of certain curves by points, where he extended the ideas of Chasles, of de Jonquières, and of Poncelet. The subject of transcendental curves he studied for a long time with great interest, and constructed a myriad of interesting patterns, but contented himself with publishing, in the joint name of himself and his pupil, the discussion of the single group of equations which he found would give the most beautiful and symmetric forms, and which he had set for his pupil to investigate.

Professor Newton was very active in securing the prompt adoption of the Metric System of Weights and Measures, both by the Connecticut Legislature and by Congress after the Conference of Nations on the subject, held in Berlin in 1863. He wrote a popular tract in 1864, giving an explanation of the system. He contributed in 1865 to the Report of the House Committee on Weights and Measures at Washington, and also to the Report of the Smithsonian Institution on this subject. He prepared an appendix consisting of these tables in form for school instruction for one of the leading arithmetics, and interested the makers of scales rulers in graduating their devices for weighing and measuring according to the Metric System. He gave his ideas to the public freely in reference to the graphical representation of all sorts of statistical information, and contributed lavishly his ideas to the authors of mathematical books used in school and college class-rooms, although he published no text-books in his own name. He was the joint author with Professor Loomis of a most elaborate paper on the climate of New Haven, which was published in the Transactions of the Connecticut Academy of Arts and Sciences. He prepared articles on the subject of Meteors for two leading cyclopædias and contributed the mathematical and astronomical definitions to Webster's International Dictionary. Professor Newton was one of the highest authorities on the subject of Life Insurance and, besides the important actuarial work which he did, computed valuable tables published by the New York Insurance Department in 1868, and later in the New Englander, a paper on the Law of Mortality that prevailed among former members of the Yale Divinity School, and still later, in Professor Dexter's Annals and Biographies, on the Length of Life of the Early Yale Graduates.

But the contributions to human knowledge, which most entitle him to fame, are those which he made on the subject of meteors, shooting stars and comets. The facts of the great star shower of 1833 had given to two New Haven men—Professors Twining and Olmstead—a clue to the true theory of the shooting stars, and this, together with the interest which the men of science at Yale had kept up in the subject of meteors, influenced Professor Newton to direct his studies towards these bodies as the time drew near for a possible recurrence of the great November shower of 1833. In 1860 he published his first paper on this subject in the *Journal of Science*, entitled "The Fireball of November 15, 1859," and this was followed by two other papers in the same journal, one on the great fireball of August 10, 1861, in which also the August group of meteors was discussed; and the other on the two fireballs of August 2 and August

6, 1860. Professor Newton had gathered a large number of observations made by persons in the localities where these bodies had attracted attention, and treated the subject with special reference to determining their nature and their velocity. Early in 1863, at the request of the Connecticut Academy, he prepared a stellar chart suited to observations at all times, which was distributed to persons at various stations for observing the August meteors. A vast amount of material was thus collected for computing the altitudes of the meteors and for obtaining some idea of their velocities. In June, 1863, Professor Newton published in the *Journal* a paper on the "Evidence of the Cosmical Origin of Shooting Stars derived from the Dates of early Star Showers," which not only established beyond question the fact that the star showers are caused by the entrance into the earth's atmosphere of bodies revolving about the sun, but gave the key to the complete solution of the problem of the November meteors. In May, 1864, he published the original accounts of thirteen remarkable displays of the November shooting stars, ranging from A. D. 902 to 1833, and in July of the same year he published a second paper in which he derived from these accounts the length of the annual period, the length of the cycle, the mean motion along the ecliptic of the node of the orbit of the group, and the length of the part of the cycle during which showers may be expected. He also showed that there were only five possible periodic times which could satisfy the observed conditions, and of these the true orbit was probably either one with a period of 354.6 days or one with a period of 33.25 years. The first of these two he thought the more likely, and computed the other elements of that orbit, but he pointed out at the same time a criterion for determining which was the true orbit when the position of the radiant should be more accurately established.

In August, 1864, Professor Newton presented to the National Academy of Sciences a comprehensive memoir on the Sporadic Shooting Stars. He had shortly before this compiled a table of computed altitudes of certain shooting stars which included substantially all that had ever been published. Using this table as a basis, he deduced the distribution of meteor paths over the sky in altitude and in azimuth, the number of shooting stars that come into our atmosphere each day, the mean length of the visible part of the meteor paths, and the number of meteoroids in the space which the earth traverses. He also deduced the remarkable fact that the mean velocity could be determined from the number of shooting stars in the different hours of the night.

These papers of Professor Newton aroused the greatest interest among mathematicians and astronomers in the subject of meteors, and especially in the star showers predicted for November, 1865 and 1866. The facts of these showers confirmed to a remarkable degree Professor Newton's theories. Leverrier and Schiaparelli, however, by independent methods showed that the period of the group was most probably 33.25 years, and Professor Adams, in 1867, by applying Professor Newton's criterion added the last link in establishing this as the true orbit of the November meteoroids.

Professor Newton, by his papers of 1863 and 1864, laid the foundation of

the Science of Meteoric Astronomy. His subsequent papers, nearly thirty in number, cover almost every topic connected with the subject. Whether in his reviews of the facts concerning the November shooting stars in the successive years from 1864 to 1869, or in the discussion of the Biela meteors of 1872 and of 1885, or in his treatment of such topics as the origin of comets, or the direct motion of comets of short period, the capture of comets by Jupiter, the effect upon the earth's velocity produced by small bodies entering the asmosphere, the relation to the earth's orbit of the former orbits of those meteorites in our collections, which were seen to fall, one prominent characteristic of his investigation was always its exhaustive character. For, whatever Professor Newton, did it was not worth the while of any one else to cover the same field.

Besides the papers which he published, his scientific activities outside the duties of his professorship were numerous and important. He organized a mathematical society in the early '60s to which he was the principal contributor, and to the successor of this society, the Yale Mathematical Club, organized in 1887, he contributed more than a score of papers. He was for many years a member of the Publishing Committee of the Connecticut Academy of Arts and Sciences. He was an associate editor of the *American Journal of Science* for thirty years. He was one of the principal founders of the Yale Observatory and practically its director till near the time of his death.

The appreciation in which his scientific ability and his labors were held is shown in the honors which he received. In 1862 he was made a member of the American Academy of Arts and Science. He was one of the original charter members of the National Academy of Sciences, founded in 1863. In 1867 he was made a member of the American Philosophical Society of Philadelphia. The degree of LL. D. was conferred upon him by Michigan University in 1868. He was made an Associate of the Royal Astronomical Society in 1872. He was Vice President of the American Association for the Advancement of Science, presiding over the section of Mathematics and Astronomy in 1875, and was President of the Association in 1885. He was made a Foreign Honorary Fellow of the Royal Society of Edinburgh in 1886, and a Foreign Member of the Royal Society of London in 1892.

At the April meeting of the National Academy in 1888 the value of Professor Newton's scientific work was publicly recognized by that body, in awarding to him the J. Lawrence Smith gold medal for his contributions to Meteoric Astronomy. His reply to the address of presentation reveals at once his modesty and his own true scientific spirit.

"Sir: I beg to express to the Academy my high appreciation of the honor you have conferred upon me. To discover some new truth in nature, even though it concerns the small things in the world, gives one of the purest pleasures in human experience. It gives joy to tell others of the treasure found. When, therefore, those best able to judge of the value of this addition to human knowledge say that it is worthy of their special public commendation, that joy is greatly increased.

I shall cherish this memorial also for that it bears the likeness of one whose true scientific spirit we all learned to admire, and whom, for his genial character, we all learned to love."

The achievements of Professor Newton, great as they were from a scientific standpoint, give no adequate idea, taken in themselves, of his power and influence. These, in a larger sense have become a part of the organic life of the University where his work was done. He built up, during a leadership of forty years, a strong and symmetrical department of Mathematics, by his comprehensive grasp of the trend of Mathematical thought, and by his wonderful power of divining the paths which lead out to fruitful fields of research, both within the domain of pure mathematics and in its applications to other sciences. Nor was the best part of his academic activities merely in his own department of studies. In moulding the general policy of the institution his counsel was invaluable; in establishing and maintaining the moral and intellectual standards, his influence was preëminent; the University bears the indelible impress of a life consecrated to the development of the noblest ideals.

Yale University.

ON THE SOLUTION OF THE QUADRATIC EQUATION.

By G. A. MILLER, Ph. D., Paris, France.

[Continued from January Number.]

The solution of the quadratic equation

$$a_0x^2+a_1x+a_2=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots A$$

is clearly equivalent to finding the two factors which are linear in x of the quantic

$$a_0x^2+a_1x+a_2.$$

When we ask whether this quantic has linear factors it is necessary to consider the domain of rationality to which we confine our attention. For illustration, we may consider the special quantic

$$x^2-4x+1.$$

If we confine ourselves to the simplest domain of rationality, viz: the domain which consists of all the rational numbers, we have to say that this quantic has no linear factors. In other words, it is irreducible in this domain. Howev-

er, if we enlarge this domain by adding to it the irrational number $\sqrt{3}$* we obtain a domain in which the quantic is clearly reducible. This domain is composed of all the numbers whose form is

$$\alpha+\beta\sqrt{3} \qquad (\alpha \text{ and } \beta \text{ being any rational numbers}).$$

According to the fundamental theorem of algebra a quantic which involves only a single variable can always be resolved into its linear factors in the domain obtained by enlarging the domain of its coefficients, if necessary, so as to include suitable new numbers. If the coefficients lie in the domain of the complex numbers the added numbers must also lie in this domain. If a quantic involves several variables it may remain irreducible even when the domain is enlarged in every possible manner.

Let x_1 and x_2 be the two roots of A. Since every rational symmetric function of the roots of an algebraic equation can be expressed rationallly in terms of its coefficients we know the value of any rational symmetric function of x_1 and x_2. This value must lie in the domain of the coefficients. In particular, we know the value of any even power of x_1-x_2. The value of the square is given by the equation

$$(x_1-x_2)^2=x_1^2+x_2^2-2x_1x_2=(x_1+x_2)^2-4x_1x_2=a_1^2-4a_0a_2/a_0^2.$$

To find the difference of the roots from the last equation we have to extract the square root of the last member. This may be impossible in the domain of the coefficients. If this domain forms a group with respect to the extraction of the square root it is clearly possible in this domain. We know that the system of ordinary complex numbers forms a group with respect to the extraction of any root. Hence we see that, if a_0, a_1, a_2 lie in the domain formed by the ordinary complex numbers, the difference of the roots of A as well as the sum of these roots must lie in the same domain.

The roots themselves may be found from these two functions by means of addition and subtraction. As any domain includes all the quantities resulting by applying these operations to any of its quantities the roots of A must also lie in the given domain of rationality. The roots may also be found by observing that their general linear function

$$\alpha x_1+\beta x_2$$

is rationally expressible as follows :†

$$\alpha x_1+\beta x_2\equiv\tfrac{1}{2}(\alpha+\beta)(x_1+x_2)+\tfrac{1}{2}(\alpha-\beta)(x_1-x_2)$$

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*By enlarging a domain of rationality by the addition of a quantity is meant the forming of the smallest domain that contains the given domain and the added quantity.

†This is an illustration of the general theorem that any rational function of the roots of an algebraic equation of degree n is rationally expressible in terms of a $n!$ valued function of the n roots.

$$\equiv \tfrac{1}{2} - (\alpha+\beta)(a_1/a_0) + (\alpha+\beta)/2a_0\sqrt{a_1{}^2 - 4a_0a_2}.$$

By letting $\alpha=1$, $\beta=0$, and $\alpha=0$, $\beta=1$ in this identity we obtain the values of x_1 and x_2 respectively.

As the ordinary complex numbers do not only form a domain of rationality but also a group* with respect to what is frequently called the most general algebraic operation, viz : that represented by

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots a_n = 0$$

(a_0, a_1, a_2, ..., a_n being ordinary complex numbers and n any posiive integer), and as they obey the commutative, distributive and associative laws of operation just like real numbers and also the law that a product cannot be zero unless one of the factors is zero, it is clear that we can reason quite generally in regard to symbols representing such numbers. It is probably largely due to this fact that other number systems are not more generally employed. In fact, no really different number system was developed until 1843. In this year Sir William Hamilton discovered and communicated to the Royal Irish Academy the system known as *Quaternions*, which is perhaps still the most important system besides that of the ordinary complex numbers. In the following year Grassmann published his *Ausdehnungslehren* in which he used a number system of a somewhat different form.

Among the investigations of later years those of Weierstrass have probably received the most attention† although important developments have been made in other directions. The fact that the ordinary complex numbers correspond to the points of a plane very naturally led to the thought that a system of higher complex numbers of the form

$$\alpha + \beta i + \gamma j \qquad (\alpha, \beta, \gamma \text{ being any real numbers})$$

might correspond to the points of space. It was easy to show that the product of two such numbers, multiplied according to the rules of ordinary numbers, may be zero when neither of the factors is zero.‡ This result naturally led to the study of numbers which do not obey all the laws of operations which the ordinary numbers obey.

The main purpose of the preceding remarks was to obtain a fairly clear view of number and of the domain of rationality as these two concepts are fundamental in the study of the solution of algebraic equations. Incidentally we indicated several methods of solving the quadratic equation A. We proceed now to consider some of the other methods of solving this equation. We shall not aim at a complete enumeration of the methods by which A may be solved. In

*It seems that Poincara was the first who considered the general number systems directly as groups. Cf. Comptes Rendus, t. 99, p. 740.

†*Gottinger Nachrichten*, 1884, page 395.

‡Cf. Harkness and Morley, *Theory of Functions*, page 8.

fact, if we would consider each modification of the operations of finding the roots of A as a new method the number of these methods would clearly be infinite. We may, for instance, form an infinite number of quantics of the form of a quadratic each of which contains the first member of A as a factor. For A may be written in the form

$$a_0x^2+a^2=a_1x.$$

Squaring both members and combining we have

$$ax^4+bx^2+c=0,$$

(a, b, c belonging to the same domain as a_0, a_1, a_2). Since the result is of the same form as A we may repeat the operation any number of times. Hence A is a factor of the quantic

$$A_0x^{2^\alpha}+A_1x^{2^{\alpha-1}}+A_2,$$

(A_0, A_1, A_2 belonging the same domain as a_0, a_1, a_2 and α being any positive integer). The roots of any one of the equations obtained by making these quantics equal zero include the roots of A. As the roots of

$$A_0y^2+A_1y+A_2=0$$

are the $2^{\alpha-1}$ powers of the roots of A it is clear that none of these transformations can simplify the solution of A. By elimination we may clearly obtain an indefinite number of additional equations containing the roots of A from the given system. In particular, if we eliminate the constant from the biquadratic equation by means of A we obtain a biquadratic equation which has the roots of A and two zero roots. Upon this elimination depends a solution recently publishin this journal. The same result might be obtained by multiplying both members of A by x^2. It is, in general, not well to raise the degree of A in the process of solution since this introduces additional roots and therefore makes the operation more complex.

Perhaps the best known method of solving A is that by which its first member is made a perfect square by the addition of the same quantity to each member. To make the quantic

$$a_0x^2+a_1x+a_2$$

a perfect square without altering its degree we may add to it the quantic

$$ax^2+bx+c$$

where two of the three numbers a, b, c are entirely arbitrary since it is only necessary that the discriminant vanishes. This idea is frequently expressed by say-

ing that the quantic to be added can be chosen in a doubly infinite number of ways. Since this quantic must also be a perfect square its own discriminant must also vanish. As this imposes another condition on its coefficients we can select the trinomial to be added to both members of A in only a simply infinite number of ways.

This number of choices might at first appear too small since in the ordinary method by which we add a constant to both members of A we apparently select both a and b arbitrarily since we let both equal zero. This would imply a doubly infinite number of choices. This apparent contradiction is explained by the fact that the vanishing of the discriminant of the added trinomial, i. e., the equation

$$b^2=4ac$$

indicates that at least two of the coefficients, including b, must be zero when one is zero. Hence the ordinary method implies that one of the coefficients of the added trinomial is selected arbitrarily and the other in accord with this equation.

To illustrate we inquire what quantics may be added to both members of the special equation

$$x^2-4x+1=0$$

so as to make both members perfect squares. Adding the given general quantic we have the equations

$$(a+1)x^2+(b-4)x+c+1=ax^2+bx+c.$$

Since the discriminants of both members must vanish we have

$$(b-4)^2=4(a+1)(c+1) \text{ and } b^2=4ac.$$

If we assign to b the arbitrary number 2 and eliminate c we have

$$a^2+a+1=0.$$

Hence a and c are the imaginary cube roots of unity, ω_1 and ω_2, and the given equation becomes*

$$-\omega_1{}^2x^2-2x-\omega_2{}^2=\omega_1x^2+2x+\omega_2$$

or

$$-1(\omega_1{}^2x^2+2x+\omega_2{}^2)=\omega_2{}^2x^2+2x+\omega_1{}^2.$$

Extracting the square root from both members we have

$$\pm i(\omega_1x+\omega_2)=\omega_2x+\omega_1$$

*It should be observed that the product of the two imaginary cube roots of unity is unity and that the square of one is equal to the other.

or $$x=\frac{\omega_1 \mp i\omega_2}{\pm i\omega_1-\omega_2}=2\pm\sqrt{3}.$$

If we let $b=4$ the first discriminant shows that one of the two factors $a+1$, $c+1$ must vanish. If we suppose that the former vanishes the given equation becomes

$$-3=-x^2+4x-4 \text{ or } x^2-4x+4=3.$$

If we suppose that the latter of the given factors vanish we obtain the equation

$$4x^2-4x+1=3x^2.$$

Instead of assigning an arbitrary value to b we might clearly assign an arbitrary value to either of the other coefficients. The simplest method is probably that in which a is made equal to zero. By making a and b equal to the corresponding coefficients with the signs changed of the equation which is to be solved and selecting c so as to satisfy the equation

$$b^2=4ac$$

we have another simple rule for completing the square. A number of other fairly convenient rules can easily be derived from what precedes.

That we can assign the given values to a and b follows from the first of the given discriminants. If we assign this value to a we determine the value of b at the same time but if we commence by assigning the given value to b neither a nor c are fully determined. We still say that the number of choices is simply infinite since a finite number multiplied into a simply infinite number is said to give a simply infinite product. The preceding remarks apply evidently also to the slight modification of the given method which consists in writing A in the form

$$a^2-b^2=0 \text{ instead of } a^2=b^2$$

and factoring the first member according to the well known formula

$$a^2-b^2=(a+b)(a-b)=(-b-a)(b-a)$$

instead of extracting the square root of the two members.

Another simple method of solving A may be described as follows: The equation A is satisfied by the affixes of two points and gives the elementary symmetric functions of these affixes. As all rational symmetric functions can be expressed rationally in terms of the elementary symmetric functions we know the affix of the middle point of the join of the roots. If the points of the plane are so transformed that this point becomes the origin the roots are the affixes of the extremities of a diameter of a circle whose center is the origin. Hence the equa-

tion in the new variable must be a pure quadratic and the solution is readily completed. If we do not assume that the coefficients are real, one root may be real while the other is imaginary. In fact the roots may be the affixes of any two points.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from January Number.]

PROPOSITION XXV. *If two straights* (*Fig.* 30.) *AX, BX existing in the same plane* (*standing upon AB, one indeed at an acute angle in the point A, and the other perpendicular at the point B*) *so always approach more to each other mutually, toward the parts of the point X, that nevertheless their distance is always greater than a certain assigned length, the hypothesis of acute angle is destroyed.*

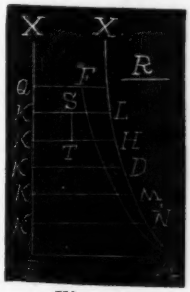


Fig. 30.

PROOF. Let R be the assigned length. If therefore in BX is assumed a certain BK any chosen multiple of the proposed length R; it follows (from the preceding Scholion) that the perpendicular erected from the point K toward the parts of AX will meet it at some point L; and again (from the present hypothesis) it follows that this KL will be greater than the aforesaid length R. Furthermore BK is understood divided into portions KK, each equal to R, even until KB is itself equal to the length R. Finally from the points K are erected to BX perpendiculars meeting AX in points L, H, D, M, even to the point N nearest the point A. Now I proceed thus.

The four angles together of the quadrilateral $KHLK$, more remote from the base AB, will be (from the preceding Proposition) greater than the four angles together of the quadrilateral $KDHK$, nearer to this base; of which quadrilateral in the same way the four angles together will be greater than the four angles together of the quadrilateral $KMDK$ subsequent toward this base. And so always even to the last quadrilateral $KNAB$, whose four angles together assuredly will be the least, in reference to the four angles together of each of the quadrilaterals ascending toward the points X.

But since are present as many quadrilaterals described in the aforesaid manner, as are, except the base AB, perpendiculars let fall from points of AX to

the straight BX; the sum of all the angles together, which are comprehended in these quadrilaterals can be reckoned. We assume that there are nine such perpendiculars let fall, and therefore so nine quadrilaterals.

We get (from Eu. I. 13) as equal to four rights the angles comprehended hither and yon at the two points of those eight perpendiculars, which lie in the middle between the base AB and the more remote perpendicular LK. So the sum of all these angles will be 32 rights.

There remain two angles at the perpendicular LK, and two at the base AB. But the angles one indeed at the point K and the other at the point B are supposed right; but the angle at the point L (from the Cor. after P. XXIII.) is obtuse. Wherefore (even neglecting the acute angle at the point A) the sum of all the angles which are comprehended by these nine quadrilaterals exceeds 35 rights. But hence follows, that the four angles together of the quadrilateral $KHLK$, more remote from the base lack less from four rights than the ninth part of one right; and that indeed even if an equal portion of the aforesaid sum of all the angles pertained to each of those quadrilaterals.

Therefore less yet will be the entered defect, since the sum of the four angles together of this quadrilateral $KHLK$ was shown the greatest of all, in relation to the four angles together of the remaining quadrilaterals.

But again; in consequence of the supposition upon which this proposition proceeds, so great a length of BK can be assumed, that as many quadrilaterals as we choose may be made on bases KK, each equal to the assigned length R.

Wherefore the defect of the four angles together of this more remote quadrilateral $KHLK$ from four rights is shown ever less both than a hundredth and than a thousandth, and thus under any assignable part of a right. Further however, LK and HK will be always (in accordance with the aforesaid supposition) greater than the designated length R. Therefore if in KL and KH are assumed KS and KT equal to KK or the length R; ST being joined, the two angles together KST, KTS will be greater, in hypothesis of acute angle, than the two angles together (from Cor. after P. XVI.) at the points H and L in the quadrilateral $THLS$, or the quadrilateral $KHLK$; and therefore (the common right angles at the points K, K being added) the four angles together of the quadrilateral $KTSK$ will be greater than the four angles together of that quadrilateral $KHLK$.

But now, since on one hand is stable and given the quadrilateral $KTSK$, in as much as constant in the given base KK, which indeed is taken equal to the assigned length R, and again constant in the two perpendiculars TK, SK equal to this base, and finally in the joining TS, which comes out completely determinate; and on the other hand the four angles together of this stable and given quadrilateral have now been shown greater than the four angles together of the quadrilateral $KHLK$ distant as far as we choose from the base AB; assuredly it follows, that the four angles together of this stable and given quadrilateral $KTSK$ are greater than any sum of angles, which lacks however you choose of being four right angles; since already it has been shown that a quadrilateral $KHLK$ can always be designated such that its four angles together shall fall short of four

rights less than any assignable part of a right. Therefore the four angles together of this stable and given quadrilateral either are equal to four rights or greater.

But then (from P. XVI.) is established the hypothesis either of right angle or of obtuse angle; and therefore (from P. V. and P. VI.) the hypothesis of acute angle is destroyed.

So is established that the hypothesis of acute angle will be destroyed, if two straights existing in the same plane so approach each other mutually ever more, that nevertheless their distance is always greater than any assigned length.

Hoc antem erat demonstrandum.

COROLLARY I. But (the hypothesis of acute angle destroyed) the controverted Pronunciatum Euclidaeum is manifest from P. 13 of this; just as that by me in this place it would be disclosed I promised in Scholion III after P. XXI of this, where we spoke of the attempt of the Arab Nassaradin.

COROLLARY II. On the other hand from this proposition, and from the preceding XXIII is manifestly gathered that not sufficient for establishing Euclidean geometry are two following points. One is: that we designate by the name of parallels those straights, which existing in the same plane possess a common perpendicular. The second indeed; that all straights existing in the same plane, of which there is no common perpendicular, and therefore which according to the assumed definition are not parallel, must, being produced toward either part ever more, somewhere meet each other, if not at a finite, at least at an infinite distance.

For again it would be requisite to demonstrate, that any two straights existing in the same plane, upon which a certain straight cutting makes two internal angles toward the same parts less than two right angles, nowhere else can receive a common perpendicular.

But that, this demonstrated, Euclidean geometry is most exactly established, will be shown below.

[To be Continued.]

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from January Number.]

XXVII. Let ABC be a triangle, right-angled at C. With O, the middle of AB, as a center, describe a circle to which either of the other sides, as BC, shall be tangent. Then,

$$BD \cdot BE = \overline{BP}^2;$$

or $(\frac{1}{2}c - \frac{1}{2}b)(\frac{1}{2}c + \frac{1}{2}b) = \frac{1}{4}a^2$. $\therefore c^2 = a^2 + b^2$.

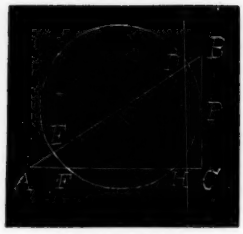


Fig. 21.

This and XVI are special cases of a more general form. For O may be any point in AB, such that the ratio of OB to AB shall be n. Our equation would then become $(nc - nb)(nc + nb) = n^2a^2$; whence, $c^2 = a^2 + b^2$.

XXVIII. Fig. 21.

Suppose $BC < AC$. Then $HC \cdot FC = \overline{PC}^2$.

But $HC \cdot FC = AF \cdot AH = AE \cdot AD = (\frac{1}{2}c - \frac{1}{2}b)(\frac{1}{2}c + \frac{1}{2}b)$, and $\overline{PC}^2 = \frac{1}{4}a^2$.

$\therefore c^2 = a^2 + b^2$.

From $BC < AC$ pass to $BC = AC$ by the theory of limits.

XXIX. Let ABC be a triangle, right-angled at C. Describe a circle, such that its center O shall be in AB, and to which the other sides shall be tangent.

Draw OD perpendicular to AB. Then,

$\overline{AT}^2 = AE \cdot AF = \overline{AO}^2 - \overline{EO}^2 = \overline{AO}^2 - \overline{TC}^2$ ..... (1).

$\overline{BP}^2 = BF \cdot BE = \overline{BO}^2 - \overline{FO}^2 = \overline{BO}^2 - \overline{CP}^2$ .... (2).

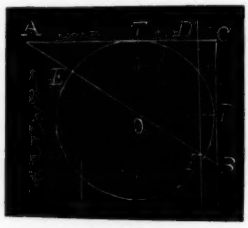


Fig. 22.

Now, $AO : OT :: AD : OD$;

$\therefore AO \cdot OD = OT \cdot AD$.

And, since $OD = OB$, $OT = TC = CP$, and $AD = AT + TD = AT + BP$,

$\therefore AT \cdot TC + CP \cdot BP = AO \cdot OB$ ....(3).

Adding (1), (2), and $2 \times (3)$,

$\overline{AT}^2 + \overline{BP}^2 + 2AT \cdot TC + 2CP \cdot BP = \overline{AO}^2 - \overline{TC}^2 + \overline{BO}^2 - \overline{CP}^2 + 2AO \cdot OB$;

$\therefore \overline{AT}^2 + 2AT \cdot TC + \overline{TC}^2 + \overline{BP}^2 + 2BP \cdot CP + \overline{CP}^2 = \overline{AO}^2 + 2AO \cdot OB + \overline{BO}^2$;

$\therefore \overline{AC}^2 + \overline{BC}^2 = \overline{AB}^2$.

XXX. Let ABC be a triangle, right-angled at C. Describe a circle, such that its center O shall be in one of the legs, as AC, and to which the other leg

and hypotenuse shall be tangent.

Then $\overline{AD}^2 = AE \cdot AC = \overline{AC}^2 - 2AC \cdot OE$;

and $\overline{BD}^2 = \overline{BC}^2$;

$\therefore$ Adding, $\overline{AD}^2 + \overline{BD}^2 = \overline{AC}^2 + \overline{BC}^2 - 2AC \cdot OE$.

$\therefore$ $\overline{AD}^2 + 2AC \cdot OE + \overline{BD}^2 = \overline{AC}^2 + \overline{BC}^2$.

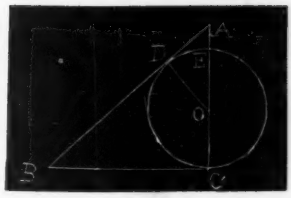


Fig. 23.

Now $AC : AC :: OD(=OE) : BC(=BD)$;

$\therefore$ $AD \cdot BD = AC \cdot OE$; $\therefore$ $\overline{AD}^2 + 2AD \cdot BD + \overline{BD}^2 = \overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

XXXI. Fig. 23.

$\overline{AD}^2 (= (AB - BD)^2) = \overline{AC}^2 - 2AC \cdot OE$,

$\therefore$ $\overline{AB}^2 - 2AB \cdot BD + \overline{BD}^2 = \overline{AC}^2 - 2AC \cdot OE$.

Adding $\overline{BD}^2 = \overline{BC}^2$, $\overline{AB}^2 - 2BD \cdot AD = \overline{AC}^2 + \overline{BC}^2 - 2AC \cdot OE$.

$\therefore$ $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

XXXII. Let ABC be a triangle right-angled at C. Draw AE parallel to BC and $= AC$. With O, the middle of AE, as a center, describe a circle, to which both AC and BC shall be tangent. Then,

$\overline{BT}^2 (= (a-b)^2) = BD \cdot BA = c(c - AD)$ .....(1).

Also, $AD : a :: 2b : c$.

$\therefore$ $AD = 2ab/c$ .....(2).

(2) in (1), $(a-b)^2 = c^2 - 2ab$. $\therefore$ $c^2 = a^2 + b^2$.

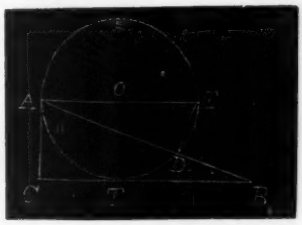


Fig. 24.

From unequal sides about the right angle pass to equal sides by the theory of limits.

[To be Continued.]

TWO DEVELOPMENTS.

By E. D. ROE, JR., Associate Professor of Mathematics in Oberlin College, Oberlin, Ohio.

[A paper read at the January meeting of the American Mathematical Society.]

It is desired to call attention here to two developments, whose statement and discussion the writer has no where met, except for $n=2$, and then not from the point of view to be suggested here. Yet it is quite possible that they may be found elsewhere.*

I. Formulas.

If $u=f(x_1, x_2, \ldots\ldots x_n)$ is a function of n variables, Δu will be used to denote the total increment in the function due to a change in all the variables, while $\Delta_{x_1, x_2 \ldots\ldots x_r} u$ will denote an increment due to a change in r variables. The two developments are as follows:

$$1. \quad u+\Delta u = u+\Delta_{x_1}u+\Delta_{x_2}u+\ldots\ldots\Delta_{x_n}u$$
$$+\Delta_{x_1}\Delta_{x_2}u+\Delta_{x_1}\Delta_{x_3}u+\ldots\ldots\Delta_{x_1}\Delta_{x_2}\ldots\ldots\Delta_{x_n}u$$

$$= u+\sum_{r=1}^{r=n}\ \sum_{\alpha_1=1}^{\alpha_1=n-r+1}\ \sum_{\alpha_2=\alpha_1+1}^{\alpha_2=n-r+2}\ldots\ldots$$
$$\sum_{\alpha_{r-1}=\alpha_{r-2}+1}^{\alpha_{r-1}=n-1}\ \sum_{\alpha_r=\alpha_{r-1}+1}^{\alpha_r=n}\alpha_r\Delta_{x_{\alpha_1}}\Delta_{x_{\alpha_2}}\ldots\ldots\Delta_{x_{\alpha_r}}u.$$

Or in determinant notation,

$$\begin{vmatrix}\Delta & 1\\ -1 & 1\end{vmatrix} u = \begin{vmatrix}\Delta_{x_1} & 1 & 1\ldots\ldots\ldots 1 & 1\\ -1 & \Delta_{x_2} & 1\ldots\ldots\ldots 1 & 1\\ -1 & -1 & \cdot\ \cdot\ \cdot\ \ \vdots & \vdots\\ \ldots\ldots & & \cdot\ \cdot\ \Delta_{x_n} & 1\\ -1 & -1 & \ldots\ldots -1 & 1\end{vmatrix} u$$

*Since the above was read Professor Fiske of Columbia, has informed the writer that "Dr. McClintock called attention to the fact, that the first result is contained in a general formula which he gave in Vol. II. page 116 of the *American Journal of Mathematics*. His formula (77) reduces in a special case to

$$\phi(E)=\phi(E_{x_1}E_{x_2}\ldots\ldots E_{x_n}) \text{ where } E=1+\Delta.$$

His operation S reduces to E when

$$\Psi=1."$$

$$=(1+\Delta_{x_1})(1+\Delta_{x_2})\ldots\ldots(1+\Delta_{x_n}u)=\prod_{r=1}^{r=n} r(1+\Delta_{x_r})u,$$

so that as operators $(1+\Delta)=\prod_{r=1}^{r=n} r(1+\Delta_{x_r})$, or the operator $1+\Delta$ is developable into the product of n operators.

2. $$u-\int du=u-\int d_{x_1}u-\int d_{x_2}u\ldots\ldots-\int d_{x_n}u+\int\int d_{x_1}d_{x_2}u+\int\int d_{x_1}d_{x_3}u+$$

$$\ldots\ldots+(-1)^n\int\int\ldots\ldots\int d_{x_1}d_{x_2}\ldots\ldots d_{x_n}u$$

$$=u+\sum_{r=1}^{r=n}\sum_{\alpha_1=1}^{\alpha_1=n-r+1}\sum_{\alpha_2=\alpha_1+1}^{\alpha_2=n-r+2}\ldots\ldots$$

$$\sum_{\alpha_r=\alpha_{r-1}+1}^{\alpha_r=n}\alpha_r(-1)^r\int\int\ldots\ldots\int d_{x_{a_1}}d_{x_{a_2}}\ldots\ldots d_{x_{a_r}}u.$$

Or in determinant notation,

$$\begin{vmatrix} -\int d & 1 \\ -1 & 1 \end{vmatrix} u=\begin{vmatrix} -\int d_{x_1} & 1 & 1\ldots. & 1 & 1 \\ -1 & -\int d_{x_2} & 1\ldots\ldots & 1 & 1 \\ -1 & -1 & & \vdots & \vdots \\ \ldots\ldots\ldots & & & -\int d_{x_n} & 1 \\ -1 & -1\ldots\ldots & & -1 & 1 \end{vmatrix} u$$

$$=(1-\int d_{x_1})(1-\int d_{x_2})\ldots\ldots(1-\int d_{x_n})u=\prod_{r=1}^{r=n} r(1-\int d_{x_r})u=0,$$

so that as operators $(1-\int d)=\prod_{r=1}^{r=n} r(1-\int d_{x_r})$, or the operator $1-\int d$ is developable into the product of n operators.

II. Proofs.

1. Let $u=f(x_1)$, then $\Delta u=\Delta_{x_1}u$, and $u+\Delta n=u+\Delta_{x_1}u=(1+\Delta_{x_1})u$.

Let $u=f(x_1x_2)$, then by the preceding, $u+\Delta_{x_2}u=(1+\Delta_{x_2})u$.

Apply the operator $1+\Delta_{x_1}$ to this equation, and

$$u+\Delta_{x_2}u+\Delta_{x_1}u+\Delta_{x_1}\Delta_{x_2}u=(1+\Delta_{x_1})(1+\Delta_{x_2})u.$$

By writing out the left member,

$$f(x_1,x_2)+f(x_1,x_2+\Delta_{x_2})-f(x_1,x_2)+f(x_1+\Delta x_1,x_2)-f(x_1x_2)+f(x_1+\Delta x_1,x_2+\Delta x_2)$$

$$-f(x_1,x_2+\Delta x_2)-f(x_1+\Delta x_1,x_2)+f(x_1,x_2)$$

$$=f(x_1x_2)+f(x_1+\Delta x_1,x_2+\Delta x_2)-f(x_1,x_2)=u+\Delta u.$$

The same process would show that by applying $(1+\Delta_{x_1})$ first, $(1+\Delta_{x_2})$ second, we would also get $u+\Delta u$. Hence would follow commutation of the two operators, so that

$$(1+\Delta_{x_1})(1+\Delta_{x_2})=(1+\Delta_{x_2})(1+\Delta_{x_1})=1+\Delta,$$

or if $u=f(x_1,x_2,\ldots\ldots x_n)$ we have proved that

$$(1+\Delta_{x_i})(1+\Delta_{x_k})=(1+\Delta_{x_k})(1+\Delta_{x_i})=1+\Delta_{x_i\,x_k},$$

i. e., any two of these operators are commutative. It follows $(n=2)$ that

$$u+\Delta_{x_2}u+\Delta_{x_1}u+\Delta_{x_1}\Delta_{x_2}u=u+\Delta_{x_1}u+\Delta_{x_2}u+\Delta_{x_2}\Delta_{x_1}u,$$

and since the ordinary addition is commutative, $\Delta_{x_1}\Delta_{x_2}u=\Delta_{x_2}\Delta_{x_1}u$, a familiar result.

Assume for $u=f(x_1,x_2\ldots\ldots x_n)$, that $u+\Delta u=(1+\Delta_{x_1}).\ \ldots.(1+\Delta_{x_n})u$.

Let $U=F(x_1,x_2\ldots\ldots x_{n+1})$.

Then by the assumption

$$U+\Delta_{x_1\ldots\ldots x_n}U=(1+\Delta_{x_1})(1+\Delta_{x_2}\ldots\ldots(1+\Delta_{x_n})U.$$

Apply $(1+\Delta_{x_{n+1}})$ to both sides of this equation remembering that we have proved commutation of operators. We get,

$$U+\Delta_{x_1\ldots\ldots x_n}U+\Delta_{x_{n+1}}U+\Delta_{x_{n+1}}\Delta_{x_1\ldots\ldots x_n}U=(1+\Delta_{x_1})\ldots\ldots(1+\Delta_{x_{n+1}})U.$$

By working out the left member, it becomes $U+\Delta U$, hence the next case takes the same form with respect to $n+1$, that the assumption had with respect to n, and since the assumption is true for $n=2$, it is true universally. The com-

mutation of any two operators brings with it the proof that $\Delta_{x_1}\Delta_{x_2}\ldots\ldots\Delta_{x_r}u$ is equal to any other one of $r!$ orders in which the r operations might be brought about. The determinant form was suggested by the formula for the development of a determinant in terms of the elements of its principal diagonal, a formula which has the same limits and number of operators in the summation. It is easily shown by adding to each column the elements of the last, when it reduces to one term, its principal diagonal term.

2. The second formula is easily shown after it has been proved for two variables. Let $u=f(x_1, x_2)$. Now u may be composed linearly of a constant, a function of x_1 alone, a function of x_2 alone, and a function of (x_1, x_2). The last must always be present, though the others may be wanting. About the constant we are not here concerned. Neglecting it,

$$u=f_1(x_1)+f_2(x_2)+\phi(x_1, x_2).$$

$$d_{x_1}u=[f'(x_1)+D_{x_1}\phi]d_{x_1},\ d_{x_2}u=[f_2'(x_2)+D_{x_2}\phi]d_{x_2}.$$

$$d_{x_1}d_{x_2}u=D_{x_1}D_{x_2}\phi dx_1dx_2. \quad \int d_{x_1}u=f_1(x_1)+\phi(x_1, x_2),$$

$$\int d_{x_2}u=f_2(x_2)+\phi(x_1, x_2). \quad \int\int d_{x_1}d_{x_2}u=\int\int D_{x_1}D_{x_2}\phi d_{x_1}d_{x_2}=\phi(x_1, x_2).$$

Hence without a constant,

$$u=\int d_{x_1}u+\int d_{x_2}u-\int\int d_{x_1}d_{x_2}u,$$

i. e., we have here complete indefinite integral of du, and

$$(1-\int d)=(1-\int d_{x_1})(1-\int d_{x_2})u=0,$$

where it is evident that there is a commutation of operators, since the ordinary additions are commutative, and $d_{x_1}d_{x_2}u=d_{x_2}d_{x_1}u$.

Assume for $u=f(x_1, x_2\ldots\ldots x_n)$, that

$$(1-\int d_{x_1})(1-\int d_{x_2})\ldots\ldots(1-\int d_{x_n})u=0. \quad \text{Let } U=F(x_1, x_2\ldots\ldots x_{n+1}).$$

Then $U=\int d_{x_1}U+\phi(x_2, x_3\ldots\ldots x_{n+1})$, where ϕ is an arbitrary function of all the variables but x_1. By the assumption

$$(1-\int d_{x_2})(1-\int d_{x_3})\ldots(1-\int d_{x_{n+1}})\phi=0, \text{ also } \phi=U-\int d_{x_1}U=(1-\int d_{x_1})U.$$

Apply the operator $(1-\int d_{x_2})\ldots\ldots(1-\int d_{x_{n+1}})$ to both sides of this equation, remembering that commutation of operators has been proved. We get

$$(1-\int d_{x_2})\ldots\ldots(1-\int d_{x_{n+1}})\phi=(1-\int d_{x_1})\ldots\ldots(1-\int d_{x_{n+1}})U=0,$$

since the left member is zero; also the last has the same form with respect to $n+1$, that the assumption had with respect to n, and since the assumption is true when $n=2$, it is universally true.

III. Applications.

1. An elegant application of the first development is its use in demonstrating that the total differential of a function is equal to the sum of its partial differentials. Let $u=f(x_1, x_2\ldots\ldots x_n)$. Then

$$\Delta u=\Delta_{x_1}u+\Delta_{x_2}u+\ldots\ldots\Delta_{x_n}u+\Sigma\Delta_{x_i}\Delta_{x_k}+\ldots\ldots\Sigma\Delta_{x_1}\Delta_{x_2}\ldots\ldots\Delta_{x_n}u.$$

Multiply through by n, where n is a number which becomes indefinitely great as the principal increment becomes indefinitely small, but so that $\lim_{\Delta x_r=0}\left(n\Delta_{x_r}u\right)$=a finite number, which is called by Hamilton, Serret, and J. M. Pierce, differential of u with respect to x_r, and may be denoted by $d_{x_r}u$. Consider $\lim_{\substack{\Delta x_r=0\\ \Delta x_k=0}}\left(n\Delta_{x_r}\Delta_{x_k}u\right)$. This equals $\lim_{\substack{\Delta x_r=0\\ \Delta x_k=0}}\left(\Delta_{x_r}n\Delta_{x_k}u\right)=\lim_{\Delta x_r=0}\Delta_{x_r}\left(d_{x_k}\right)=0$.

A *fortiori*, zero will be the limit of any term of higher order. We have then at once by taking the limits of both members,

$$du=d_{x_1}+d_{x_2}u+\ldots\ldots d_{x_n}u.$$

2. The second formula gives us the complete solution of a differential equation, when it is a perfect differential. The solution is,

$$u=\int d_{x_1}u+\int d_{x_2}u+\ldots\ldots\int d_{x_n}u-\sum\int\int d_{x_i}d_{x_k}u+\sum\int\int\int d_{x_i}d_{x_k}d_{x_j}u\ldots.$$

$$+(-1)^{n-1}\int\int\ldots\ldots\int d_{x_1}d_{x_2}\ldots\ldots d_{x_n}u.$$

To this a constant may be added which is determined as usual by corresponding values of the variables and function. It may or may not be zero. When the function is such that farther differentiation of the first differentials will cut them down rapidly, this formula ought to be practically useful. When $n=2$, we can by this formula solve the problem of finding the orthogonal and isothermal curves to a given system, $u=c$, when u satisfies the equation $D_x{}^2u+D_y{}^2u=0$.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

73. Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, New Jersey.

A man owes me \$100 due in 2 years, and I owe him \$200 due in 4 years. When can I pay him \$100 to settle the account equitably, money being worth 6%, and the interest to draw interest until the time of settlement?

Solution by FREDERIC R. HONEY, New Haven, Connecticut.

One dollar placed at 6% compound interest, in two years will amount to $1.06^2=\$1.1236$. Therefore the present value of \$100.00 due in 2 years is $\$100.00\div 1.1236=\$89.00$ *very nearly*.

One dollar placed at 6% compound interest in four years will amount to $1.06^4=\$1.2625$. Therefore the present value of \$200.00 due in 4 years is $\$200.00\div 1.2625=\$158.416$.

Therefore the difference between \$158.416 and \$89.00=\$69.416, is the amount of my debt to A at the present time.

Since \$1.00 placed at 6% compound interest in 6 years will amount to $1.06^6=\$1.4185$, \$69.416 at the same rate will, in 6 years, amount to $69.416\times 1.4185=\$98.4666$.

And since the simple interest on one dollar for 1 year is \$0.06, the simple interest on \$98.4666 is $98.4666\times 0.06=\$5.908$ for one year. Therefore the interest $\$100.00-\$98.4666=\$1.5334$ will accrue in $1.5334\div 5.908=0.2594$ years.

And $6.+0.2594=6.2594$ the number of years hence when \$100.00 should be paid, in order to settle the account equitably.

J. M. Bandy sent solutions of Nos. 71 and 72 too late for credit in February number.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

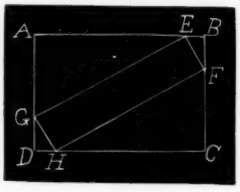
SOLUTIONS OF PROBLEMS.

67. Proposed by F. M. PRIEST, St. Louis, Mo.

Required: The length of a piece of carpet that is a yard wide with square ends, that can be placed diagonally in a room 40 feet long and 30 feet wide, the corners of the carpet just touching the walls of the room.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas; P. S. BERG, Larimore, North Dakota; J. SCHEFFER, A. M., Hagerstown, Maryland; J. M. COLAW, A. M., Monterey, Virginia; R. H. WAGONER, Westerville, Ohio; J. F. YOTHERS, Westerville, Ohio; J. T. FAIRCHILD, Crawfis College, Ohio; CHAS. C. CROSS, Laytonsville, Maryland; and O. S. WESTCOST, Chicago, Illinois.

Let $AB=40=a$, $BC=30=b$, $EF=3=c$, $BF=x$, $BE=y$.



$\therefore\ x^2+y^2=c^2$ .....(1).

From the triangles CHF and BEF we get $HC : CF=BF : BE$ or $a-y : b-x=x : y$.

$\therefore\ ay-y^2=bx-x^2$ .....(2).

(1) in (2) gives $bx-x^2=a\sqrt{c^2-x^2}-c^2+x^2$.

$\therefore\ 4x^4-4bx^3+(a^2+b^2-4c^2)x^2+2bc^2x+c^4-a^2c^2=0.$

$\therefore\ 4x^4-120x^3+2464x^2+540x-14319=0.$

$\therefore\ x=2.43372+,\ y=1.75414+.$

$HG=\{(a-y)^2+(b-x)^2\}^{\frac{1}{2}}=47.14494+.$

II. Solution by COOPER D. SCHMITT, Professor of Mathematics, University of Tennessee, Nashville, Tennessee; W. H. HARVEY, Portland, Maine; and B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

In the figure used above, let $AB=DC=a=40$ feet, the length of the room.

$AD=BC=b=30$ feet, the width of the room.

$EF=GH=c=3$ feet, the width of the carpet.

Let $x=HF=GE$, the length of the carpet, and the angle CFH=the angle $HGD=\theta$.

Then $x\sin\theta=CF$, $c\sin\theta=DH$, $x\cos\theta=HC$, and $c\cos\theta=DG$.

$\therefore\ c\sin\theta+x\cos\theta=DC=a$ .....(1),

and $x\sin\theta+c\cos\theta=BC=b$ .....(2).

Multiplying (1) by (2), and collecting, we get

$cx(\sin^2\theta+\cos^2\theta)+(x^2+c^2)\sin\theta\cos\theta=ab$, or

$$cx+(x^2+c^2)\sin\theta\cos\theta=ab \quad \text{.....(3).}$$

Squaring (1) and (2) and adding the results, we get

$$c^2+x^2+4cx\sin\theta\cos\theta=a^2+b^2 \quad \text{.....(4).}$$

From (3), $\sin\theta\cos\theta=(ab-cx)/(x^2+c^2)$. Substituting this value of $\sin\theta\cos\theta$ in (4) and reducing, we get,

$$x^4-(a^2+b^2+2c^2)x^2+4abcx-c^2(a^2+b^2-c^2)=0 \quad \ldots\ldots\ldots\ldots(5).$$

Restoring numbers in (5), we have

$$x^4-2518x^2+14400x-22419=0.$$

Solving this equation by Horner's Method, we find $x=47.145$ feet, nearly.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Virginia. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

58. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

A line passes through a fixed point and rotates uniformly about this point. Another line passes through a point which moves uniformly along the arc of a given curve and rotates uniformly about this point. Develop a method for finding the locus of intersection of these two lines. Apply to case of circle and straight line.

II. Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

Let O be the origin, P_3 the fixed point, its coördinates being (r_3, θ_3), and let AB be a given position of line through P_3. Let $P_1(r_1, \theta_1)$ be position of point on curve and CD the line through it, both corresponding to the position AB of other line. Also let HK be position of AB revolved through an $\angle\psi$, and let $P_2(r_2, \theta_2)$ and EF be the corresponding position of P_1 and CD.

Let $r=f(\theta)$ be equation to curve P_1P_2. Let η=the angle made by AB, and η_1 the one made by CD with a polar axis. Let a=angular rate of revolution of AB, and na of CD.

$\therefore$ $\angle$ between CD and $EF=n\psi$.

Let b=linear rate of movement of P_1. Then $\psi/a=P_1P_2/b$ ..... (1).

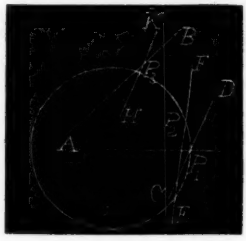
Equation to KH is $r=[r_3\sin(\eta+\psi-\theta_3)]/\sin(\eta+\psi-\theta)$ .....(2).

Equation to EF is $r=(r_2\sin(\eta_1+n\psi-\theta_2)]/\sin(\eta_1+n\psi-\theta)$.....(3).

By integration,

$$P_1P_2=\int_{\theta_1}^{\theta_2}\sqrt{r^2+[(dr/d\theta)]^2}\,d\theta=\int_{\theta_1}^{\theta_2}\sqrt{[f(\theta)]^2+[f'(\theta)]^2}\,d\theta\ldots\ldots\ldots\ldots\ldots(4),$$

which gives P_1P_2 in terms of θ_2, θ_1 being known. Then substitute from (4) in (1) to get ψ in terms of θ_2. Substitute this value, and also $f(\theta_2)$ for r_2 in (2) and (3). Then by eliminating (θ_2) we have resulting the equation to the locus of the intersecting of the lines. The solution depends on our ability to integrate (4). Now if the given lines are not straight, it is evident that the only changes are in equations (2) and (3). These may be derived from the equations to the lines in original position by a method of transformation of coördinates. For example, the equation to HK may be derived from that to AB by revolving the pole and polar axis about P_3 through an angle equal to ψ and in the opposite direction. If the given curve is a circle and the lines straight, the problem can be definitely solved as follows:



Transform coördinates so that center of circle shall be pole and OP_1 the the polar axis. Then $r=f(\theta)$ becomes $r=c$.

Let $(r_1, \theta_1)(r_2, \theta_2)(r_3, \theta_3)$ represent the new coördinates of points P_1, P_2, P_3, respectively.

Then $r_1=r_2=c$. $\theta_1=0$. $P_1P_2=c\theta_2$. From (1) $\psi=ac\theta_2/b$.

(2) becomes $r=[r_3\sin(\eta+ac\theta_2/b-\theta_3)]/\sin(\eta+ac\theta_2/b-\theta)\ldots\ldots\ldots\ldots(5)$.

(3) becomes $r=[c\sin(\eta_1+nac\theta_2/b-\theta_2)]/\sin(\eta_1+nac\theta_2/b-\theta)\ldots\ldots\ldots(6)$.

(5) can be solved for θ_2 and the result can be substituted in (6), giving the equation required. Then if desired the coördinates can be again transformed to the original form.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

43. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

Two weights P and Q rest on the concave side of a parabola whose axis is horizontal, and are connected by a string, length l, which passes over a smooth peg at the focus, F. [*Bowser's Analytical Mechanics*, page 54.]

I. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi; COLMAN BANCROFT, M. Sc., Professor of Mathematics, Hiram College, Hiram, Ohio; GEORGE LILLEY, LL. D., Portland, Oregon; G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas, and the PROPOSER.

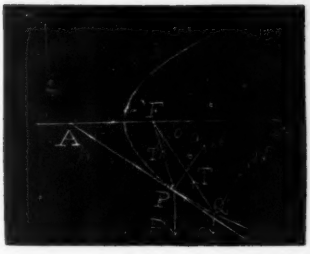
Let AX be the horizontal axis of the parabola, F the focus, P' and Q' the positions of the weights P and Q, T the tention of the string $P'FQ'$, N and N' the normal reactions at P' and Q' respectively, θ and θ' the angles between the axis AX and the focal radii to P' and Q' respectively, A the intersection of the axis and the tangent at P'. Denote the latus rectum by $4m$.

$$\angle\theta = \angle FAP' + \angle FP'A = 2\angle FAP',$$

by property of parabola.

$\therefore \angle FAP' = \frac{1}{2}\theta$.

Since N is inclined to the vertical at the same angle that the tangent is inclined to the horizontal, we have for equilibrium of the forces at P', resolving vertically and horizontally,



$N\cos\frac{1}{2}\theta + T\sin\theta = P$, and $N\sin\frac{1}{2}\theta = T\cos\theta$,

from which

$T(\cot\frac{1}{2}\theta\cos\theta + \sin\theta) = P$, or, $T\cot\frac{1}{2}\theta = P$.

Similarly, $T\cot\frac{1}{2}\theta' = Q$. $\therefore \cot\frac{1}{2}\theta = P/Q\cot\frac{1}{2}\theta'$.

From the polar equation of a parabola,

$$FP' = \frac{2m}{1-\cos\theta}, \quad FQ' = \frac{2m}{1-\cos\theta'}.$$

But $FP' + FQ' = l$. $\therefore \frac{2m}{1-\cos\theta'} = l - \frac{2m}{1-\cos\theta'}$, $\cos\theta' = 1 - \frac{2m(1-\cos\theta)}{l(1-\cos\theta)-2m}$,

$$\sin^2\tfrac{1}{2}\theta' = \frac{m(1-\cos\theta)}{l(1-\cos\theta)-2m}, \quad \cot\tfrac{1}{2}\theta' = \sqrt{\frac{l(1-\cos\theta)+m\cos\theta-3m}{m(1-\cos\theta)}}.$$

Then, $\cot\frac{1}{2}\theta = \frac{P}{Q}\sqrt{\frac{l(1-\cos\theta)+m\cos\theta-3m}{m(1-\cos\theta)}}$. Since $1-\cos\theta = \frac{2}{\cot^2\frac{1}{2}\theta+1}$, the preceding equation gives $\cot\frac{1}{2}\theta = \frac{P\sqrt{l-2m}}{\sqrt{m(P^2+Q^2)}}$.

II. Solution by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

If, in the figure above, θ represents the angle XFP, and θ_1, XFQ, then

$$FP = r = \frac{2m}{1-\cos\theta} = \frac{m}{\sin^2\frac{1}{2}\theta}. \quad \therefore \sin^2\tfrac{1}{2}\theta = \frac{m}{r}, \text{ and } \cos^2\tfrac{1}{2}\theta = \frac{r-m}{r}.$$

Since $FQ=l-r$, $\sin^2\frac{1}{2}\theta_1=\frac{m}{l-r}$, and $\cos^2\frac{1}{2}\theta_1=\frac{l-r-m}{l-r}$.

Since the tension is the same in all parts of the string and the angle between the radius vector and tangent is half the angle between the radius vector and the X axis, $T=P\tan\frac{1}{2}\theta=Q\tan\frac{1}{2}\theta_1$.

$$\therefore \frac{P}{Q}=\frac{\operatorname{ctn}\frac{1}{2}\theta}{\operatorname{ctn}\frac{1}{2}\theta_1}, \quad \therefore \frac{P^2}{P^2+Q^2}=\frac{r-m}{l-2m}=\frac{m\operatorname{ctn}^2\frac{1}{2}\theta}{l-2m},$$

$$\therefore \operatorname{ctn}\tfrac{1}{2}\theta=\frac{P\sqrt{l-2m}}{\sqrt{m(P^2+Q^2)}}.$$

III. Solution by **WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.**

Let r', r'' be the parts of the string l joining the focus and the weights P and Q; θ and θ' the angles which r' and r'' make with the axis of X.

For the equilibrium of P and Q, resolving along the tangents through P and Q, T being the tension in the string,

$$T\sin\tfrac{1}{2}\theta=P\cos\tfrac{1}{2}\theta \quad\ldots\ldots(1). \qquad T\sin\tfrac{1}{2}\theta'=Q\cos\tfrac{1}{2}\theta' \quad\ldots\ldots(2).$$

These give $P/Q\cot\frac{1}{2}\theta=\cot\frac{1}{2}\theta'$ .....(3).

The equations to the curve are

$$r'=\frac{2m}{1+\cos\theta}, \quad r''=\frac{2m}{1+\cos\theta'};$$

then $r'+r''=2m\left(\frac{1}{1+\cos\theta}+\frac{1}{1+\cos\theta'}\right)=l$ .....(4).

Now $\cos\theta=\frac{\cot^2\frac{1}{2}\theta-1}{\cot^2\frac{1}{2}\theta+1}$, $\cos\theta'=\frac{\cot^2\frac{1}{2}\theta'-1}{\cot^2\frac{1}{2}\theta'+1}$ .....(5).

(3) and (5) and the resulting values of $\cos\theta$ and $\cos\theta'$ in (4) and reducing gives

$$\frac{2P^2\cot^2\frac{1}{2}\theta+(P^2+Q^2)}{2P^2\cot^2\frac{1}{2}\theta}=\frac{l}{2m} \quad\ldots\ldots(6).$$

Subtracting unity from both members of (6) and taking the square root of the result,

$$\tan\tfrac{1}{2}\theta=\frac{P}{\sqrt{P^2+Q^2}}\sqrt{\frac{l-2m}{m}} \quad\ldots\ldots(7).$$

In which put $\pi-\theta$ for θ for Bowser's result.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

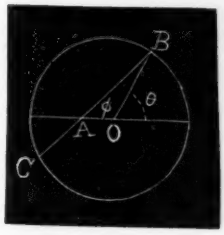
SOLUTIONS OF PROBLEMS.

43. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

In a circle whose radius is a, chords are drawn through a point distant b from the center. What is the average length of such chords, (1), if a chord is drawn from every point of the circumference, and (2), if they are drawn through the point at equal angular intervals?

Solution by the PROPOSER.

In the figure, let BC represent the chord passing through the point A whose distance from O is $OA=b$. Put $BC=x$, $\angle BOA=\theta$, $\angle BAO=\phi$, A_1=first average required, and A_2=second average required.



Then $x=2(a^2-b^2\sin^2\phi)^{\frac{1}{2}}$.

Hence, $A_1=1/\pi\int_0^{\pi}x\,d\theta$. From triangle AOB,

$a\sin(\theta+\phi)=b\sin\phi$. $\therefore\ \theta+\phi=\sin^{-1}(b/a\sin\phi)$.

$$\therefore\ d\theta=-d\phi+\frac{b\cos\phi\,d\phi}{(a^2-b^2\sin^2\phi)}.$$

$$\therefore\ A_1=2/\pi\int_0^{\pi}[(a^2-b^2\sin^2\phi)^{\frac{1}{2}}-b\cos\phi]d\phi=\frac{4a}{\pi}\ E\left(\frac{b}{a},\ \tfrac{1}{2}\pi\right).$$

$$A_2=1/\pi\int_0^{\pi}x\,d\phi=\frac{4a}{\pi}\int_0^{\frac{1}{2}\pi}[1-(b^2/a^2)\sin^2\phi]^{\frac{1}{2}}\,d\phi=\frac{4a}{\pi}\ E\left(\frac{b}{a},\ \tfrac{1}{2}\pi\right).$$

$\therefore\ A_1=A_2$. If $b=0$, $A_1=A_2=2a$. If $b=a$, $A_1=A_2=4a/\pi$.

This problem was also solved by G. B. M. Zerr. His solution will appear in the next issue.

44. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

What is the average length of all the chords that may be drawn from one extremity of the major axis of an ellipse if they are drawn at equal angular intervals?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas; J. SCHEFFER, A. M., Hagerstown, Maryland, and the PROPOSER.

Let r=length, e=eccentricity of ellipse, θ=angle the chord makes with major axis. Then

$$r=\frac{2a(1-e^2)\cos\theta}{1-e^2\cos^2\theta}.\qquad \Delta=\text{average length}=\frac{\int_0^{\frac{1}{2}\pi}r\,d\theta}{\int_0^{\frac{1}{2}\pi}d\theta}.$$

$$\therefore \Delta=\frac{4a(1-e^2)}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\cos\theta d\theta}{1-e^2\cos^2\theta}. \quad \therefore \Delta=\frac{4a\sqrt{1-e^2}}{\pi e}\tan^{-1}\frac{e}{\sqrt{1-e^2}}.$$

When $e=\theta$, $\Delta=\frac{4a}{\pi}$, since $\frac{1}{e}\tan^{-1}\frac{e}{\sqrt{1-e^2}}=1$.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Virginia. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

42. Proposed by E. B. ESCOTT, 6123 Ellis Avenue, Chicago, Illinois.

To find a triangle whose sides and median lines are commensurable.

Solution by J. W. TESCH, in "L'Intermediare des Mathematiciens" for October, 1896. Translated and adapted by J. M. COLAW, A. M., Monterey, Virginia.

Suppose the sides to be $2a$, $1+2b-b^2+\frac{1}{4}a^2$, $1-2b-b^2+\frac{1}{4}a^2$; then we will have $m_1=\pm(1+b^2-\frac{1}{4}a^2)$,

$$m_2{}^2=\tfrac{1}{2}[4a^2+(1-2b-b^2+\tfrac{1}{4}a^2)^2]-\tfrac{1}{4}(1+2b-b^2+\tfrac{1}{4}a^2)^2,$$

$$\text{or } m_2{}^2=\tfrac{1}{64}a^4+\tfrac{1}{8}(17-6b-b^2)a^2+\tfrac{1}{4}(1-12b+2b^2+12b^3+b^4).$$

In order that the second member may be a perfect square, it is necessary that $\frac{1}{64}(17-6b-b^2)^2=4\times\frac{1}{64}\times\frac{1}{4}(1-12b+2b^2+12b^3+b^4)$, whence $2b=3$.

Thus the sides become $2a$, $\frac{7}{4}+\frac{1}{4}a^2$, $-\frac{17}{4}+\frac{1}{4}a^2$, or

(1) $16a$, $2(a^2+7)$, $\pm2(a^2-17)$; $m_1=\pm2(a^2-13)$, $m_2=a^2+23$.

We will have $m_3{}^2=a^4+190a^2-191$. The values of a, which render the second number a perfect square, are 1, 3, 5, ... ; $m_3=0$, 40, 72, ... ; but none of these values satisfy (1); therefore, after the method of Euler, (*Vollständige Anleitung zur Algebra*, or the French translation by J.-G. Garnier, 2 vols., with the additions of Lagrange, Paris, 1807), it is necessary to proceed as follows:

By supposing $a=3+h$, we may write

$$(3+h)^4+190(3+h)^2-191=(40+ph+h^2)^2,$$

where p is a coefficient to be determined. Developing, we have

$$1248+244h+12h^2=80p+(80+p^2)h+2ph^2.$$

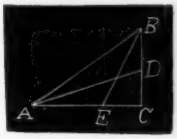
If we take $80p=1248$ or $p=15+3/5$, we find $h=-(4+2/15)$, which gives $a^2=(17/15)^2$. We also have for the three sides, after some easy reductions: 510, 466, 884, and for the medians 659, 683, 208. This is perhaps the simplest case in whole numbers.

43. Proposed by H. C. WILKES, Skull Run, West Virginia.

To find, if possible, a right angled triangle, the bisectors of the acute angles of which, can be expressed by integral whole numbers.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let ABC be a right triangle, right angled at C, AD the bisector of $\angle A$, and BE the bisector of $\angle B$.



Put $BC=a$, $AC=b$, $AB=c$, $DC=a_1$, $EC=b_1$, $EB=c_1$, and $AD=c_2$. Then $BD=a-a_1$, and $AE=b-b_1$. From geometrical relations we obtain $a^2+b^2=c^2$ .....(1); $c_1^2=ac-b_1(b-b_1)$.....(2); $b-b_1 : b_1=c : a$.....(3).

From (3) we get $b : b_1=c+a : a$; whence $b_1=ab/(c+a)$, and $b-b_1=bc/(c+a)$.

$\therefore c_1^2=ac-ab^2c/(c+a)^2=ac-ac(c^2-a^2)/(c+a)^2=2a^2c/(c+a)$.

By a similar process, we find $c_2^2=2b^2c/(c+b)$.

From (1), $c^2-b^2=a^2$, or $(c+b)(c-b)=a^2$. Put $c+b=tp^2$ and $c-b=tq^2$, t, p, and q being any values. Then $a=tpq$, $b=t(p^2-q^2)/2$, and $c=t(p^2+q^2)/2$. Whence $c_1^2=2t^2p^2q^2(p^2+q^2)/(p+q)^2$, and $c_2^2=t^2(p^2-q^2)^2(p^2+q^2)/4p^2$.

When $p^2+q^2=\square$, $c_2^2=\square$, and $c_1^2=2\times\square$. When $p^2+q^2=2\times\square$, $c_2^2=2\times\square$, and $c_1^2=\square$.

$\therefore$ *Both* bisectors cannot be rational; *one* of them will be $\sqrt{2}$ times a number, when the *other* is a rational whole number.

II. Solution by the PROPOSER.

Let bx, by, and $x+y$ be, respectively, the sides and base of a right angled triangle, and let x and y be the greater and less segments of the base cut by the bisector. Then the bisector will be $\sqrt{y^2(b^2+1)}$ and if the bisector be integral, b^2+1 must$=\square$. b must therefore be an improper fraction, and will always be the quotient of the sum of the other two sides divided by the bisected side.

Now let CAB be a triangle, and let $AB=x^2+y^2$, $CA=x^2-y^2$ and $CB=2xy$, $CA+AB/CB=x/y$. $(x/y)^2+1$ may be a square, but $AB+CB/CA=(x+y)/(x-y)$. $[(x+y)/(x-y)]^2+1$ will be a multiple of the $\sqrt{2}$ and cannot be a square.

$\therefore$ If a rational right angled triangle have an integral bisector of one of its acute angles, the bisector of the other acute angle must be a multiple of $\sqrt{2}$ and and cannot be integral.

[Remark.—On page 155, Vol. II. of the MONTHLY, we have, when the sides are 59.4107, 47.4072, 35.8067, the bisectors 40 and 50. It is doubtful whether the sides and bisectors both can be integral. ZERR.]

44. Proposed by P. S. BERG, Principal of Schools, Larimore, North Dakota.

Two trees whose heights are 40 and 80 feet, respectively, stand on opposite sides of a stream 30 feet wide. What path does a squirrel take in leaping from the top of the higher to the top of the lower? What is the length of the path?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

The path is a parabola. Let the top of the higher tree be origin, "across the river" positive, v=velocity, β=angle of projection, then the equation is $y=\tan\beta x-gx^2/(2v^2\cos^2\beta)$, in which we must know either v, or β. Substituting $x=30$, $y=-40$ we get

$$v^2=45g/(4\cos^2\beta+3\sin\beta\cos\beta) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$S=\frac{1}{v^2\cos^2\beta}\int_0^{30}\sqrt{v^4\cos^4\beta+(gx-v^2\sin\beta\cos\beta)^2}\,dx.$$

Let $v^2\cos^2\beta=a$, $gx-v^2\sin\beta\cos\beta=y$, $30g-v^2\sin\beta\cos\beta=y_1$, $-v^2\sin\beta\cos\beta=-y_2$.

$$\therefore S=\frac{1}{ag}\int_{-y_2}^{y_1}\sqrt{a^2+y^2}\,dy$$

$$=\frac{1}{2ag}\{y_1\sqrt{a^2+y_1^2}+y_2\sqrt{a^2+y_2^2}\}+\frac{a}{2g}\log[(\sqrt{a^2+y_1^2}+y_1)/(\sqrt{a^2+y_2^2}-y_2)]$$

Let $\beta=45°$; then $v^2=90g/7$, $y_1=165g/7$, $y_2=45g/7=a$.

$\therefore S=\frac{5}{14}(11\sqrt{130}+9\sqrt{2})+\frac{45}{14}\log[(\sqrt{130}+11)/(3\sqrt{2}-3)]$.

Let $\beta=0$; then $v^2=45g/4$, $y_1=30g$, $y_2=0$, $a=45g/4$.

$\therefore S=5\sqrt{73}+\frac{45}{8}\log[(\sqrt{73}+8)/3]$.

NOTES.

INTERNATIONAL CONGRESS OF MATHEMATICIANS AT ZURICH IN 1897.

"It is known that the idea of an international congress of mathematicians has been, above all in these latter days, the object of numerous deliberations on the part of scientists interested in its realization. It has appeared to them, by reason of the excellent results obtained in other scientific domains by an international 'entente', that assuring the execution of this project would have very weighty advantages.

As outcome of a very active exchange of views, accord was reached on a

prime point. Switzerland, by its central geographic situation, by its traditions and its experience of international congresses, appeared designated to invite a first attempt at a reunion of mathematicians. In consequence Zurich is chosen as seat of the Congress.

The mathematicians of Zurich do not disguise from themselves the difficulties they will have to surmount. But in the interest of this enterprise, they have thought it their duty not to decline the overtures so flattering that have been made them from all sides. They have decided therefore to take all preparatory measures for the future congress and, to the extent of their powers, to contribute to its success. So, with the concurrence of mathematicians of other nations, was formed the undersigned committee of organization, charged *to bring together at Zurich in* 1897 *the mathematicians of the entire world.*

The congress, in which you are cordially invited to take part, will take place at Zurich the 9, 10 and 11 of August, 1897, in the halls of the federal polytechnic school. The committee will not fail to communicate to you, in time opportune, the text of the program determined, begging you to inform them of your adherence. But even at present it may be said that the scientific works and questions of administration will pertain to subjects of general interest or recognized importance.

Scientific congresses have also this precious advantage, to favor and keep up personal relations. The local committee will not fail to accord all its solicitude to this part of its task, and, with this aim, it will elaborate a modest programme of fêtes and intimate reunions.

May the hopes reposed in this first congress be fully realized! May numerous participants contribute by their presence to create, among colleagues, not alone coherent scientific relations, but also cordial bonds based on personal acquaintance! Finally, may our congress serve the advancement and the progress of the mathematical sciences!''

The invitation of which the above is a translation is signed by eleven from Zurich and ten associates, as committee.

Readers of the AMERICAN MATHEMATICAL MONTHLY already know the persistent efforts of Vasiliev of Kazan and Laisant of Paris to establish this congress.

It is matter for rejoicing that their noble endeavors have been crowned with this definite success. GEORGE BRUCE HALSTED.

THE SAME OLD BLUNDER.

In the *Nation* of November 26, 1896, in a review of Cajori's History of Elementary Mathematics, the reviewer himself makes a blunder so appalling that it should not go unnoticed.

He says Cajori ''does not name Prof. J. J. Littrow of Vienna, whose demonstration is yet worth notice. Littrow proves first that the three angles of a triangle are $=2R$. Thus: When a side α and angles BC are given, angle A is determined; it is $=F(\alpha BC)$; and as an angle may be viewed as an abstract number, it has no relation to one measure in space: angle $A=F(BC)$ simply,'' etc.

Now where has the *Nation's* reviewer been buried not to know that this very pseudo-proof was given by Legendre, and its fallacy shown by George Paxton Young in the *Canadian Journal* for November, 1856, forty years ago, and again in the *Canadian Journal* for July, 1860, pages 356-358?

GEORGE BRUCE HALSTED.

Austin, Texas.

James Joseph Sylvester, the great mathematician, Sivilian professor of geometry at Oxford, formerly professor of mathematics at the Johns Hopkins University, and in 1841 at the University of Virginia, died in London on March 15th, aged eighty-three years. Also the eminent mathematician Dr. Karl Weierstrass, died at Berlin on February 19th, aged eighty-one years. In the death of these two men, mathematics sustains a great loss. Both did much to broaden and deepen mathematical knowledge. Sylvester has written much on invariants, the theory of equations, theory of partitions, multiple algebra, the theory of numbers, the theory of reciprocants, etc., while Weierstrass has given special attention to the theory of functions of a complex variable. For a biography of Sylvester, by Dr. Halsted, see MONTHLY, Vol. I., No. 9. B. F. F.

BOOKS.

Composite Geometrical Figures. By George A. Andrews, A. M. 63 pages. Price 55 cents. Ginn and Company, Boston, and London. 1896.

The figures in this little book are constructed for the demonstration of more than one proposition. Ten of the figures are designed for review work, while the last general figure is intended for re-review work of all the theorems of plane geometry. Under each figure are illustrative demonstrations, followed by series of easy examples which require the pupil to apply the general principles of geometry to the specific conditions of the figures. It will be seen that the book is not designed to take the place of other text-books, but is intended to be used with them for reviews and for supplemental easy original work. The plan of the work is rather unique, and it will be useful to teachers who feel the need in their classes for the specific application of geometrical principles. J. M. C.

National Geographic Monographs: (1) *Physiographic Processes*, by J. W. Powell; (2) *Physiographic Regions of the U. S.*, by J. W. Powell; (3) *Lakes and Sinks of Nevada*, by I. C. Russell; (4) *Mt. Shasta*, by J. S. Diller. Price 20 cents each, or $1.50 for a set of ten. American Book Company, New York, Cincinnati, and Chicago.

These monographs on the physical features of the earth's surface furnish fresh and interesting material with which to supplement the regular text books. They have been written with exceptional care and ability and are not only very serviceable for such use, but are very interesting to the general reader as well. J. M. C.